A Comparison of the EKF, SPKF, and the Bayes Filter for Landmark-Based Localization

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Outline

- Background
- Objective
- Experimental Setup
- Results
- Discussion
- Conclusion

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Background

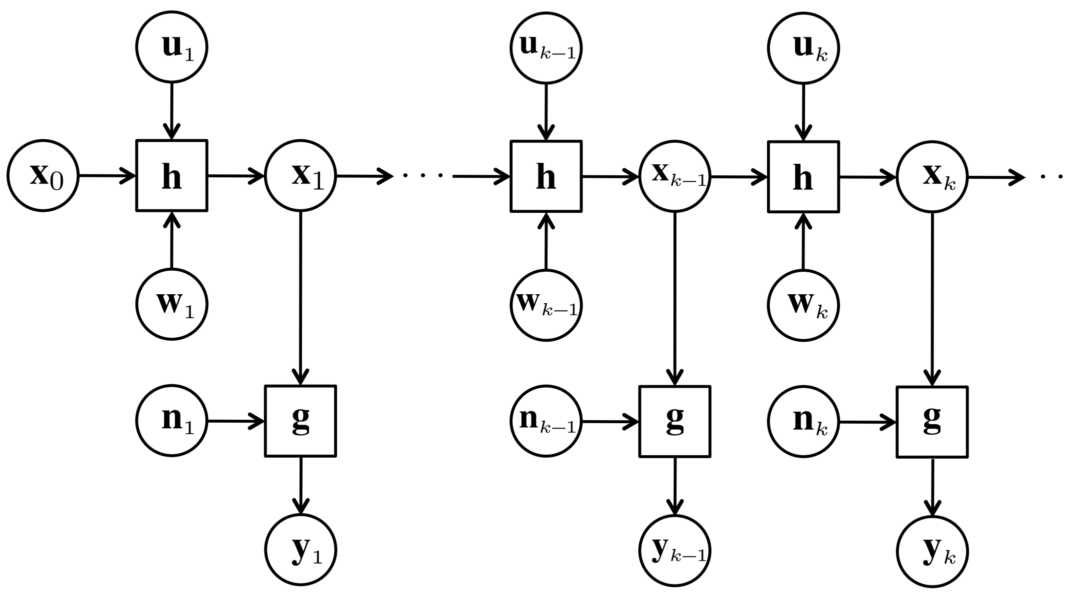
- What is state estimation?
  - The problem of *estimating* the *state* of a process
  - Noisy measurements
  - Sensor fusion
- Sample applications
  - Robot localization and mapping
  - Chemical process control
  - Weather prediction

State Space Models

- Discrete-time, nonlinear models:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{h}\left(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k\right) \\ \mathbf{y}_k &= \mathbf{g}\left(\mathbf{x}_k, \mathbf{n}_k\right) \end{aligned}$$

- Markov assumption:



Bayes Filter

$$\underbrace{p(\mathbf{x}_k|\mathbf{u}_{1:k},\mathbf{y}_{1:k})}_{\substack{\text{posterior}\\ \text{belief}}} = \eta \underbrace{p(\mathbf{y}_k|\mathbf{x}_k)}_{\substack{\text{observation}\\ \text{correction}\\ \text{using } \mathbf{g}(\cdot)}} \int \underbrace{p(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{u}_k)}_{\substack{\text{motion}\\ \text{prediction}\\ \text{using } \mathbf{h}(\cdot)}} \underbrace{p(\mathbf{x}_{k-1}|\mathbf{u}_{1:k-1},\mathbf{y}_{1:k-1})}_{\substack{\text{prior}\\ \text{belief}}} \, d\mathbf{x}_{k-1}$$

- Intractable
  - Infinite-dimensional
  - Infinite computation time

- Kalman Filter assumptions:
  - [Kalman, 1960]
  - Gaussian state PDF

$$p\left(\mathbf{x}_k \middle| \mathbf{u}_{1:k}, \mathbf{y}_{1:k}\right) \rightarrow \mathcal{N}\left(\hat{\mathbf{x}}_k, \hat{\mathbf{P}}_k\right)$$

  - Zero-mean, Gaussian noise PDFs:

$$\mathbf{w}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_k\right), \quad \mathbf{n}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}_k\right)$$

Infinite Computation

- Extended Kalman Filter (EKF)
  - [Schmidt, 1967]
  - Conventional approach involving linearization

- Sigma-Point Kalman Filter (SPKF)
  - [Julier et al., 1995]
  - Modern approach of passing selective samples through the nonlinearity

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Literature Review

- [Julier and Uhlmann, 1997]
  - Simulated spacecraft reentry tracking problem
  - Found that the SPKF estimate had a lower mean-squared error compared to ground truth than the EKF

- [van der Merwe and Wan, 2004]
  - Simulated and real GPS/INS UAV guidance problem
  - SPKF had a lower RMS error compared to ground truth than the EKF with similar computational requirements

Novel Contribution

- A comparison of the performance of the EKF and SPKF as approximations to the Bayes Filter, in the context of a real-world nonlinear state estimation problem

Methodology

- Implement algorithms
  - EKF
  - SPKF
  - Bayes Filter (via Monte Carlo Sampling)

$$p(\mathbf{x}_k|\mathbf{u}_{1:k},\mathbf{y}_{1:k}) = \eta\, p(\mathbf{y}_k|\mathbf{x}_k)\int p(\mathbf{x}_k|\mathbf{x}_{k-1},\mathbf{u}_k)p(\mathbf{x}_{k-1}|\mathbf{u}_{1:k-1},\mathbf{y}_{1:k-1})d\mathbf{x}_{k-1}$$

- Real-world nonlinear state estimation problem
  - Indoor mobile rover localization problem

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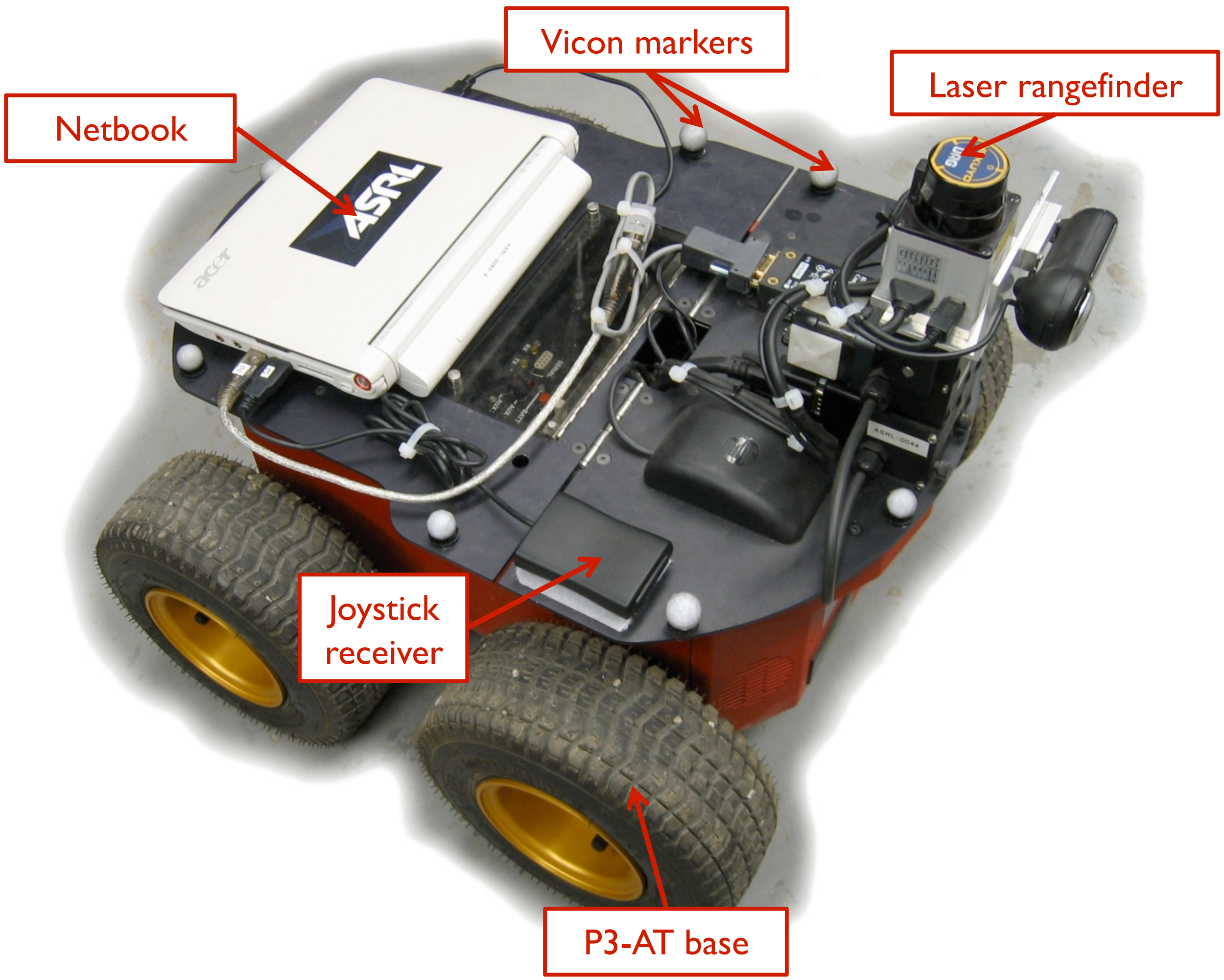
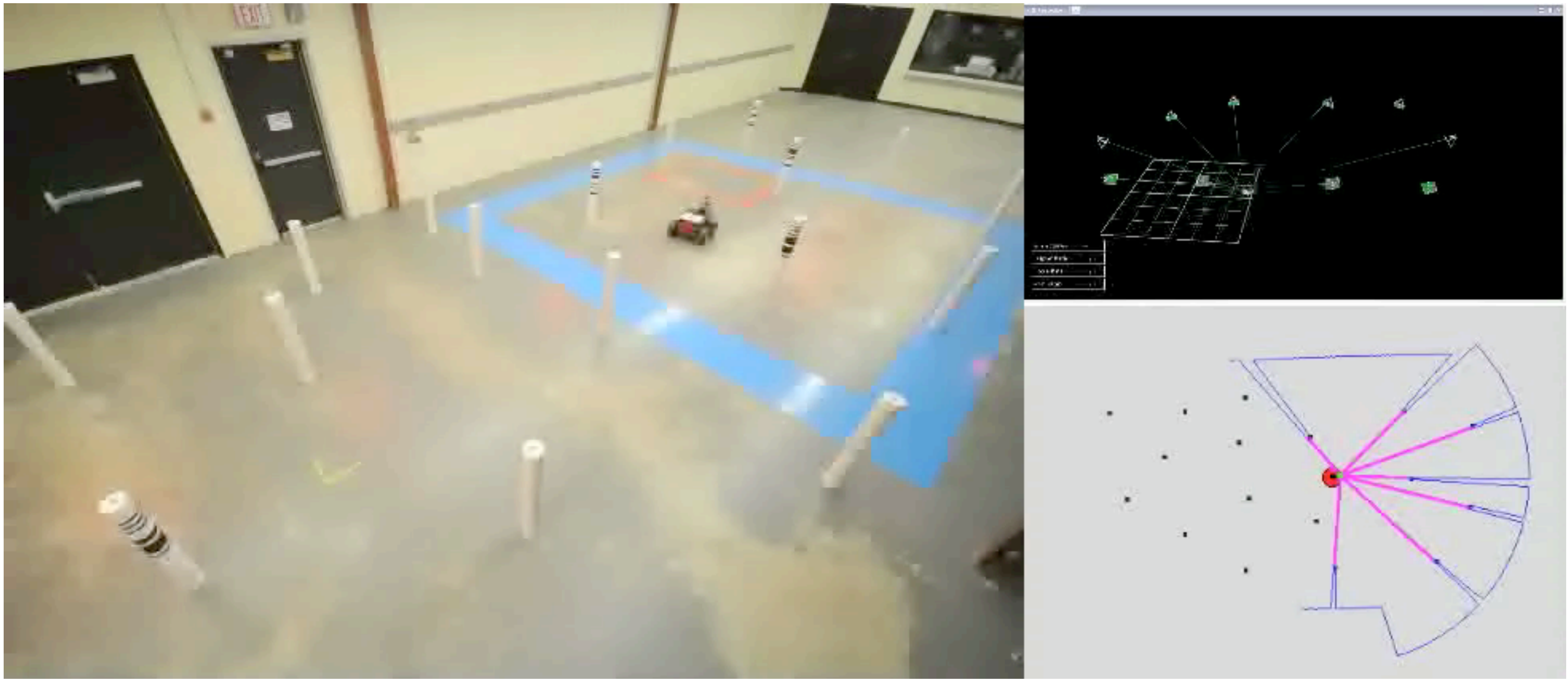
Pioneer3-AT


Vicon markers
Laser rangefinder
Netbook
Joystick receiver
P3-AT base

| Vicon System

Lab Space

Experimental Setup | # Video



Rover Traverse

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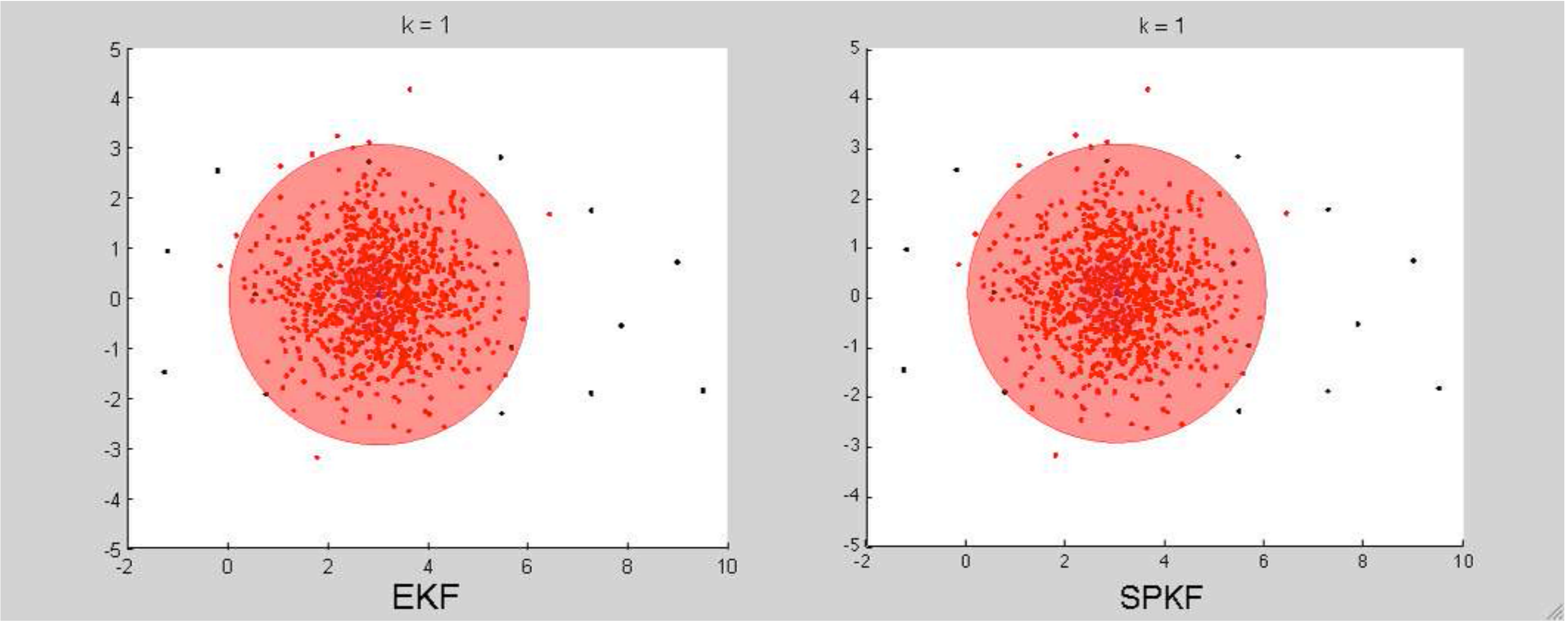
Estimation Video



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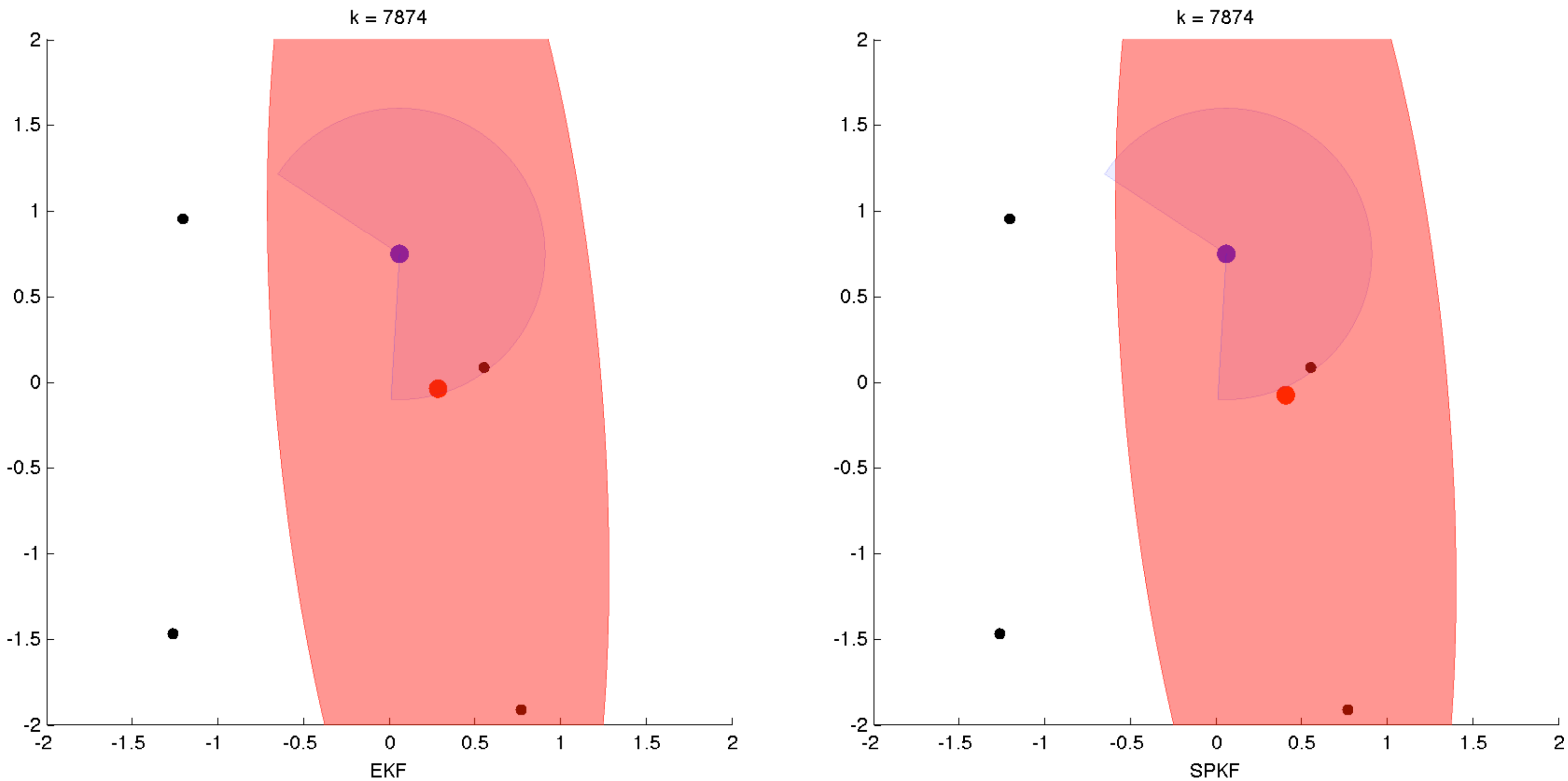
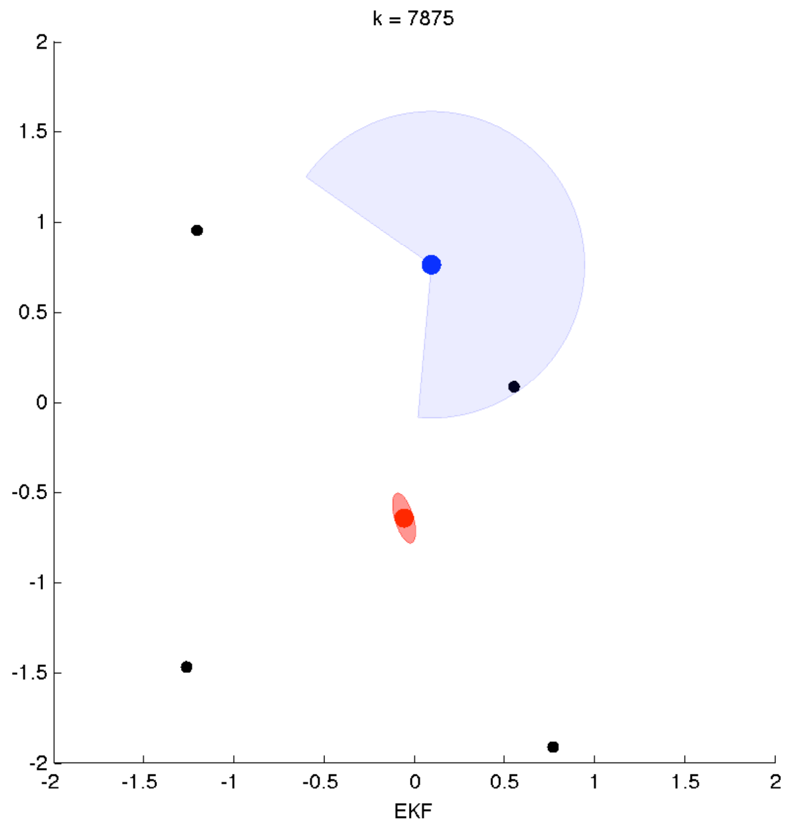
Error Spike Example

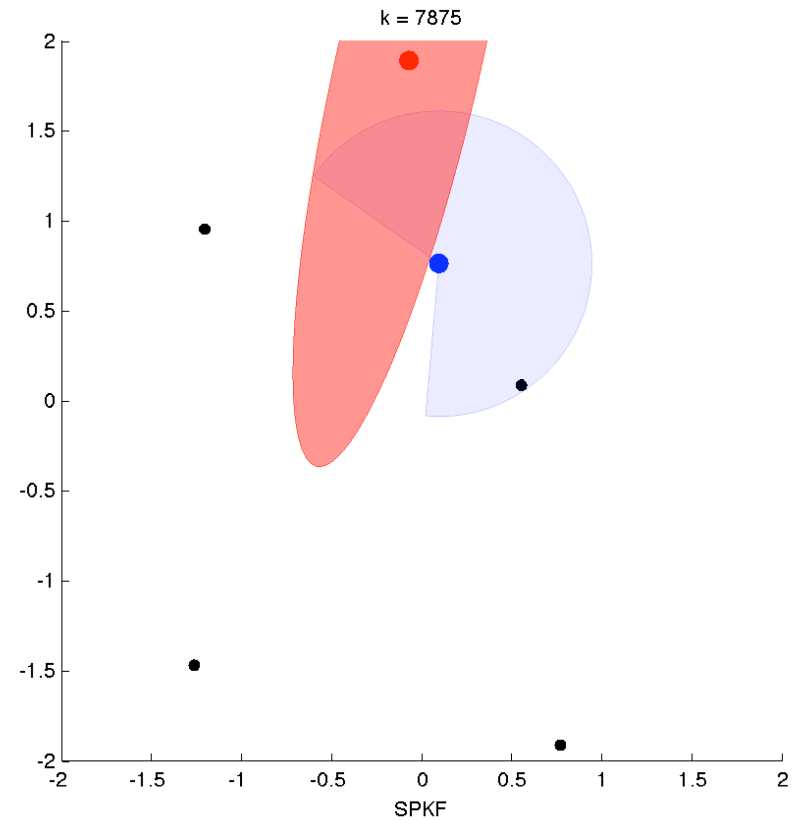


EKF SPKF

Error Spike Example

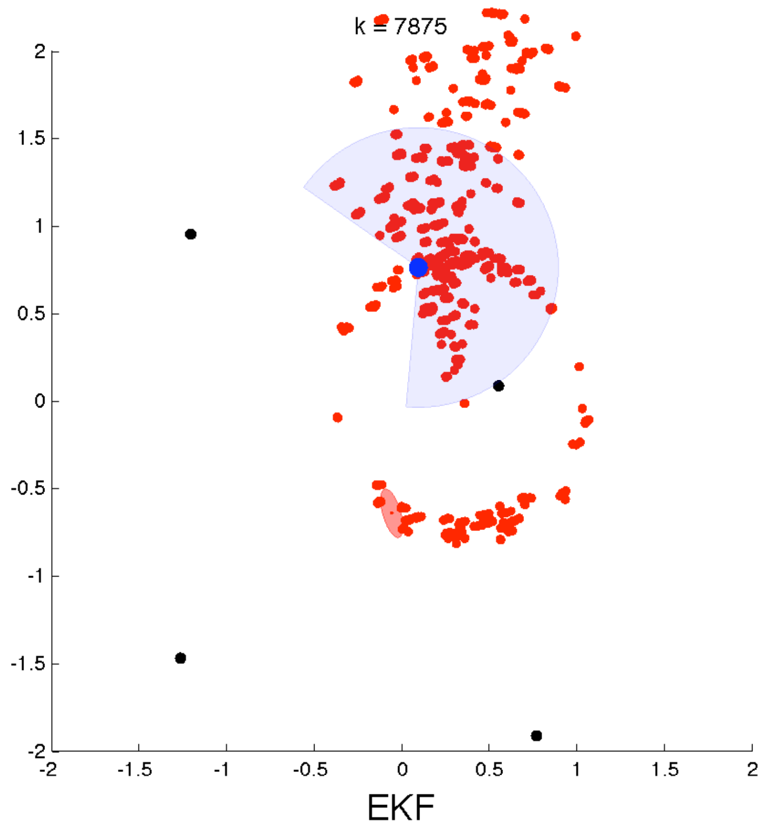


EKF

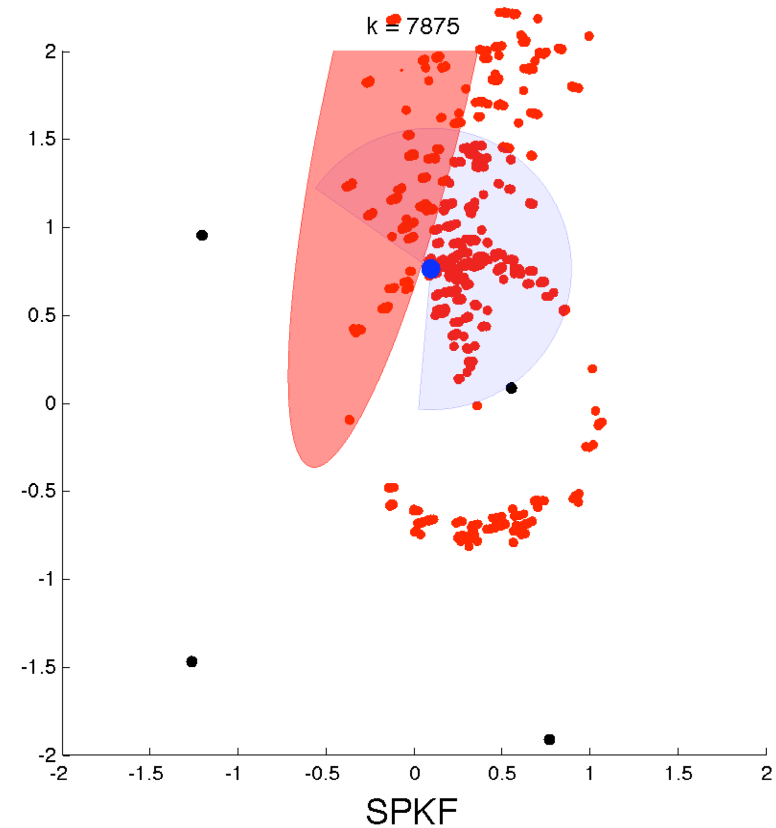


SPKF

Discussion |
Error Spike Example



EKF



SPKF

Recursive Estimation

$$p(\hat{\mathbf{x}}_k|\mathbf{u}_{1:k},\mathbf{y}_{1:k}) = \eta\, p(\mathbf{y}_k|\hat{\mathbf{x}}_k)\int p(\hat{\mathbf{x}}_k|\hat{\mathbf{x}}_{k-1},\mathbf{u}_k)p(\hat{\mathbf{x}}_{k-1}|\mathbf{u}_{1:k-1},\mathbf{y}_{1:k-1})d\hat{\mathbf{x}}_{k-1}$$

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Conclusion

- SPKF is easy to implement
- SPKF outperforms the EKF (in this problem)
  - Handling of nonlinearities
  - Approximating the Bayes Filter
- However, the Recursive Bayesian approach is fundamentally flawed

References

- [Kalman, 1960]
  - R. E. Kalman. A new approach to linear filtering and prediction problems. Journal of Basic Engineering, 1960.
- [Schmidt, 1966]
  - S. F. Schmidt. Applications of state space methods to navigation problems. In Advanced Control Systems, volume 3, pages 293-340. Academic Press, 1966.
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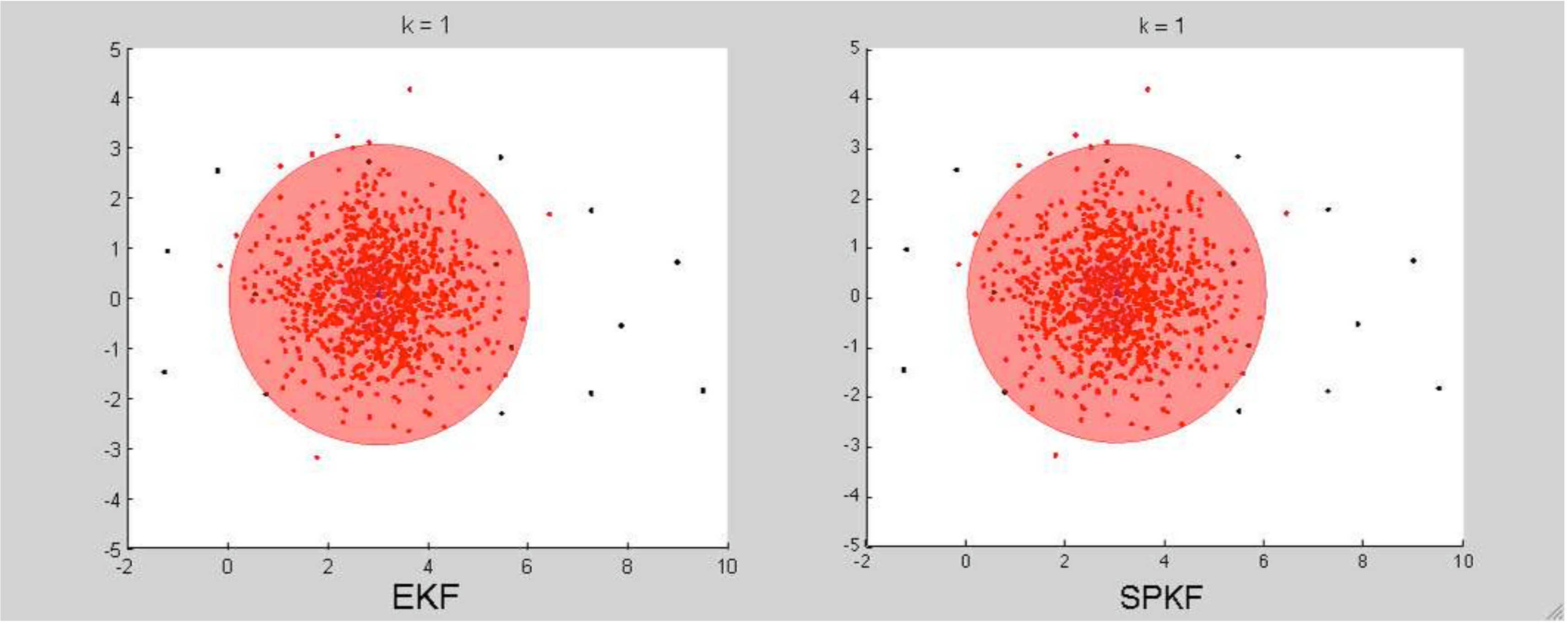
  - S.J. Julier, J.K. Uhlmann and H.F. Durrant-Whyte. A new approach for filtering nonlinear systems. In the proceedings of the American Control Conference, pages 1628-1632, 1995.
- [Julier and Uhlmann, 1997]
  - S. J. Julier and J. K. Uhlmann. A new extension of the Kalman Filter to nonlinear systems. Int. Symp. Aerospace/Defense Sensing, Simul. and Controls, 1997.
- [van der Merwe and Wan, 2004]
  - R. van der Merwe and E. Wan. Sigma-Point Kalman Filters for integrated navigation. In Proceedings of the 60th Annual Meeting of The Institute of Navigation (ION), 2004.
- [Vicon]
  - Vicon MX. http://www.vicon.com/products/viconmx.html.

Questions?



$$
\begin{aligned}
\hat{\mathbf{P}}_k^- &= \mathbf{H}_{\mathbf{x},k}\hat{\mathbf{P}}_{k-1}\mathbf{H}_{\mathbf{x},k}^T + \mathbf{H}_{\mathbf{w},k}\mathbf{Q}_k\mathbf{H}_{\mathbf{w},k}^T \\
\hat{\mathbf{x}}_k^- &= \mathbf{h}\left(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_k, \mathbf{0}\right) \\
\mathbf{K}_k &= \hat{\mathbf{P}}_k^-\mathbf{G}_{\mathbf{x},k}^T\left(\mathbf{G}_{\mathbf{x},k}\hat{\mathbf{P}}_k^-\mathbf{G}_{\mathbf{x},k}^T + \mathbf{G}_{\mathbf{n},k}\mathbf{R}_k\mathbf{G}_{\mathbf{x},k}^T\right)^{-1} \\
\hat{\mathbf{P}}_k &= \left(\mathbf{1} - \mathbf{K}_k\mathbf{G}_{\mathbf{x},k}\right)\hat{\mathbf{P}}_k^- \\
\hat{\mathbf{x}}_k &= \hat{\mathbf{x}}_k^- + \mathbf{K}_k\left(\mathbf{y}_k - \mathbf{g}\left(\hat{\mathbf{x}}_k^-, \mathbf{0}\right)\right)
\end{aligned}
$$

SPKF Prediction

$$
\begin{aligned}
\mathcal{X}_{i,k}^{-} &:= \mathbf{h}\left(\mathcal{X}_{i,k-1}, \mathbf{u}_k\right) \\
\hat{\mathbf{x}}_k^{-} &:= \frac{1}{L+\kappa}\left(\kappa \mathcal{X}_{i,k}^{-} + \frac{1}{2}\sum_{i=1}^{2L} \mathcal{X}_{i,k}^{-}\right) \\
\hat{\mathbf{P}}_k^{-} &:= \frac{1}{L+\kappa}\left(\kappa\left(\mathcal{X}_{0,k}^{-} - \hat{\mathbf{x}}_k^{-}\right)\left(\mathcal{X}_{0,k}^{-} - \hat{\mathbf{x}_k^{-}}\right)^T + \frac{1}{2}\sum_{i=1}^{2L}\left(\mathcal{X}_{i,k}^{-} - \hat{\mathbf{x}}_k^{-}\right)\left(\mathcal{X}_{i,k}^{-} - \hat{\mathbf{x}}_k^{-}\right)^T\right)
\end{aligned}
$$

SPKF Correction

$$
\begin{aligned}
\mathcal{Y}_{i,k} &:= \mathbf{g}\left(\mathcal{X}_{i,k}^{-}, \mathcal{N}_{i,k}\right) \\
\hat{\mathbf{y}}_k &:= \frac{1}{L+\kappa}\left(\kappa \mathcal{Y}_{i,k} + \frac{1}{2}\sum_{i=1}^{2L} \mathcal{Y}_{i,k}\right) \\
\mathbf{V}_k &:= \frac{1}{L+\kappa}\left(\kappa\left(\mathcal{Y}_{0,k} - \hat{\mathbf{y}}_k\right)\left(\mathcal{Y}_{0,k} - \hat{\mathbf{y}}_k\right)^T + \frac{1}{2}\sum_{i=1}^{2L}\left(\mathcal{Y}_{i,k} - \hat{\mathbf{y}}_k\right)\left(\mathcal{Y}_{i,k} - \hat{\mathbf{y}}_k\right)^T\right) \\
\mathbf{U}_k &:= \frac{1}{L+\kappa}\left(\kappa\left(\mathcal{X}_{0,k}^{-} - \hat{\mathbf{x}}_k^{-}\right)\left(\mathcal{Y}_{0,k} - \hat{\mathbf{y}}_k\right)^T + \frac{1}{2}\sum_{i=1}^{2L}\left(\mathcal{X}_{i,k}^{-} - \hat{\mathbf{x}}_k^{-}\right)\left(\mathcal{Y}_{i,k} - \hat{\mathbf{y}}_k\right)^T\right) \\
\mathbf{K}_k &:= \mathbf{U}_k \mathbf{V}_k^{-1} \\
\hat{\mathbf{x}}_k &:= \hat{\mathbf{x}}_k^{-} + \mathbf{K}_k\left(\mathbf{y}_k - \hat{\mathbf{y}}_k\right) \\
\hat{\mathbf{P}}_k &:= \hat{\mathbf{P}}_k^{-} - \mathbf{K}_k \mathbf{U}_k^T
\end{aligned}
$$

Particle Filter (PF)

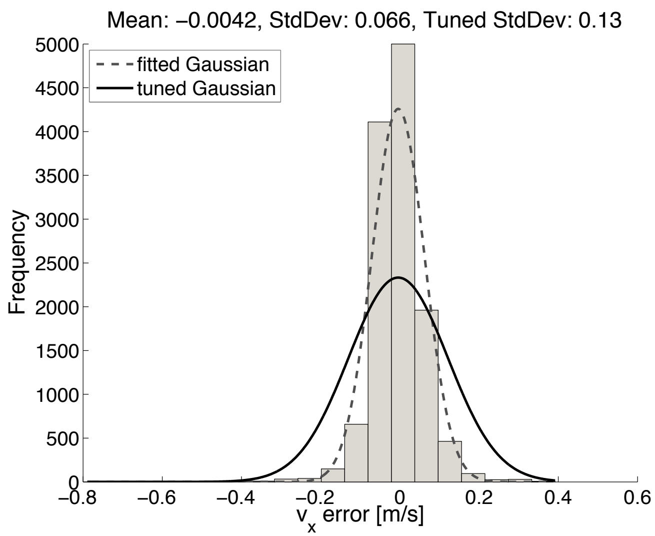
$$
\begin{aligned}
\mathbf{x}_{k-1}^{(m)} &\leftarrow p\left(\mathbf{x}_{k-1} \middle| \mathbf{u}_{1:k-1}, \mathbf{y}_{1:k-1}\right) \\
\mathbf{w}_{k}^{(m,l_m)} &\leftarrow p(\mathbf{w}_k) \\
\mathbf{x}_{k}^{(m,l_m)^-} &:= \mathbf{h}\left(\mathbf{x}_{k-1}^{(m)}, \mathbf{u}_k, \mathbf{w}_l^{(m,l_m)}\right), \quad (\forall m, l_m) \\
w_{k}^{(m,l_m)} &:= p\left(\mathbf{y}_k \middle| \mathbf{g}\left(\mathbf{x}_k^{(m,l_m)^-}, \mathbf{0}\right)\right) \\
\mathbf{x}_{k}^{(m,l_m)} &\leftarrow \left\{\mathbf{x}_k^{(m,l_m)^-}, w_k^{(m,l_m)}\right\}
\end{aligned}
$$

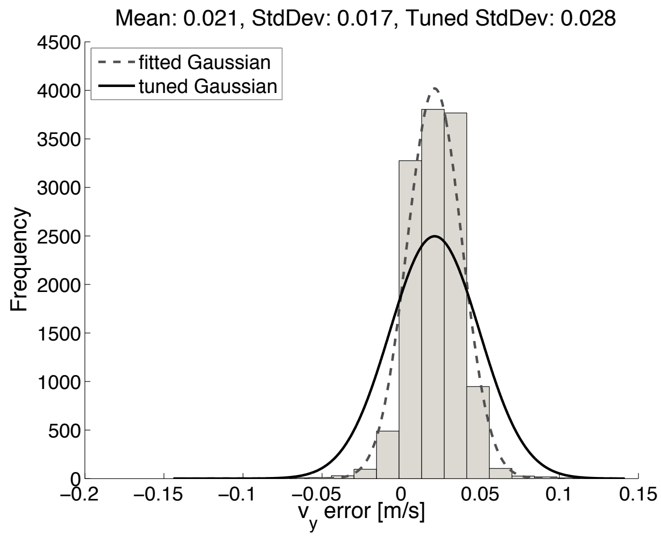
Process Model

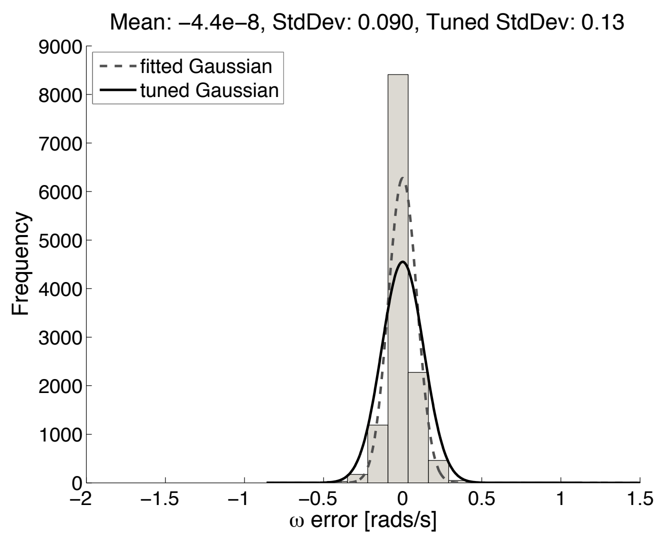
$$\underbrace{\begin{bmatrix} x_k \\ y_k \\ \theta_k \end{bmatrix}}_{\mathbf{x}_k} = \underbrace{\underbrace{\begin{bmatrix} x_{k-1} \\ y_{k-1} \\ \theta_{k-1} \end{bmatrix}}_{\mathbf{x}_{k-1}} + T \begin{bmatrix} \cos\theta_{k-1} & -\sin\theta_{k-1} & 0 \\ \sin\theta_{k-1} & \cos\theta_{k-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\underbrace{\begin{bmatrix} v_{x_k} \\ v_{y_k} \\ \omega_k \end{bmatrix}}_{\mathbf{u}_k} + \mathbf{w}_k \right)}_{\mathbf{h}(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k)}$$

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$$

Process Noise

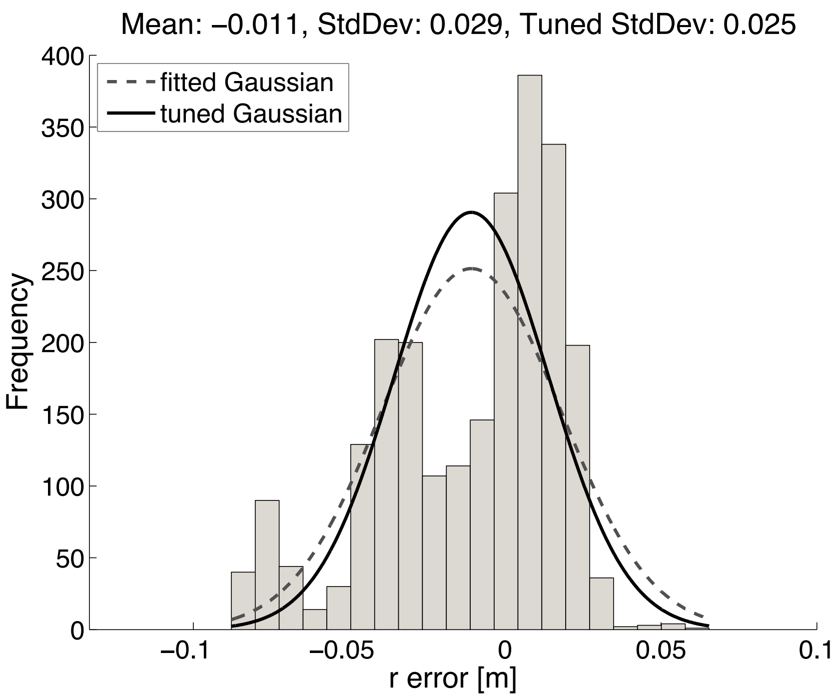


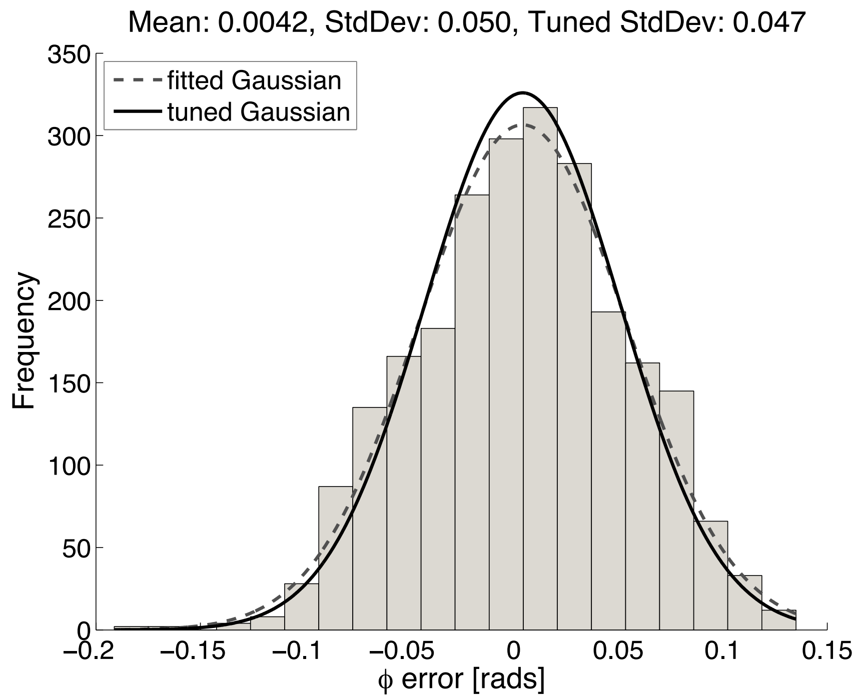




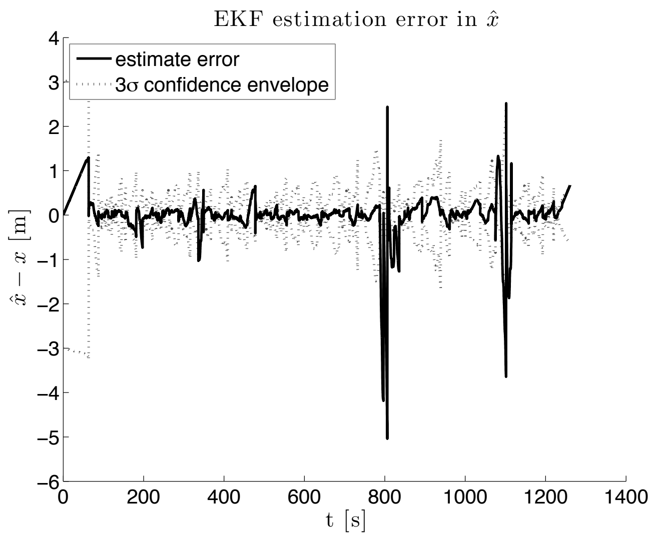
$$\underbrace{\begin{bmatrix} r_{k,l} \\ \phi_{k,l} \end{bmatrix}}_{\mathbf{y}_{k,l}} = \underbrace{\begin{bmatrix} \sqrt{\left(x_l - x_k - d\cos\theta_k\right)^2 + \left(y_l - y_k - d\sin\theta_k\right)^2} \\ \operatorname{atan2}\left(y_l - y_k - d\sin\theta_k, x_l - x_k - d\cos\theta_k\right) - \theta_k \end{bmatrix} + \mathbf{n}_{k,l}}_{\mathbf{g}_l(\mathbf{x}_k, \mathbf{n}_{k,l})}$$

$$\mathbf{n}_{k,l} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}\right)$$

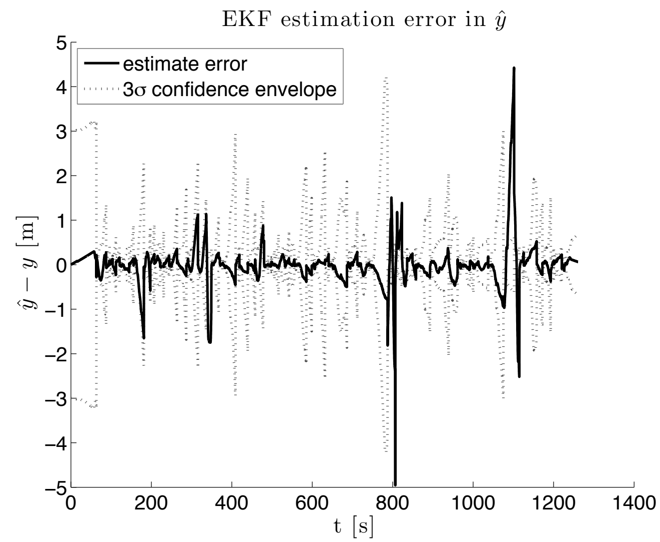

Mean: −0.011, StdDev: 0.029, Tuned StdDev: 0.025
fitted Gaussian
tuned Gaussian
Frequency
r error [m]


Mean: 0.0042, StdDev: 0.050, Tuned StdDev: 0.047
fitted Gaussian
tuned Gaussian
Frequency
ϕ error [rads]

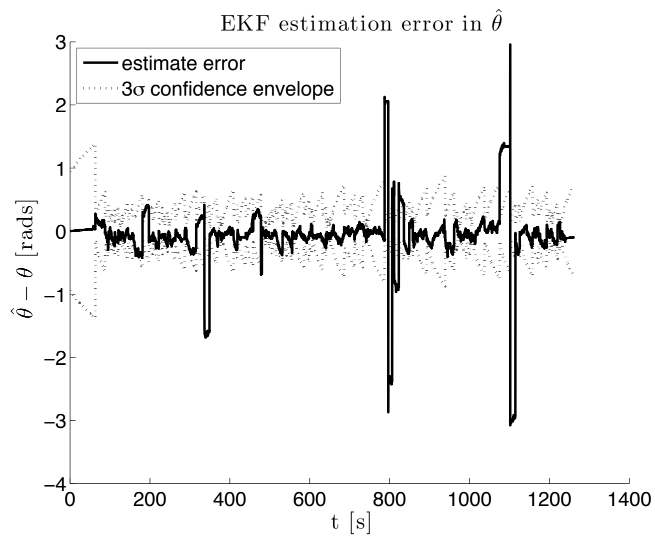
Results | EKF



EKF estimation error in $\hat{x}$
estimate error
3σ confidence envelope
$\hat{x} - x$ [m]
t [s]

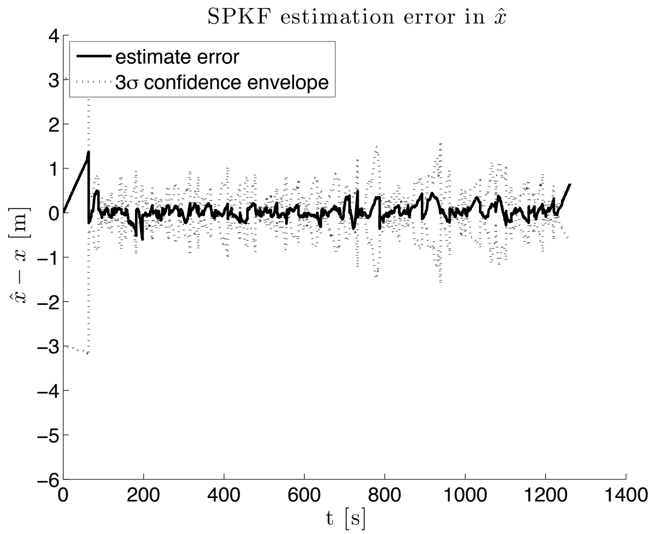


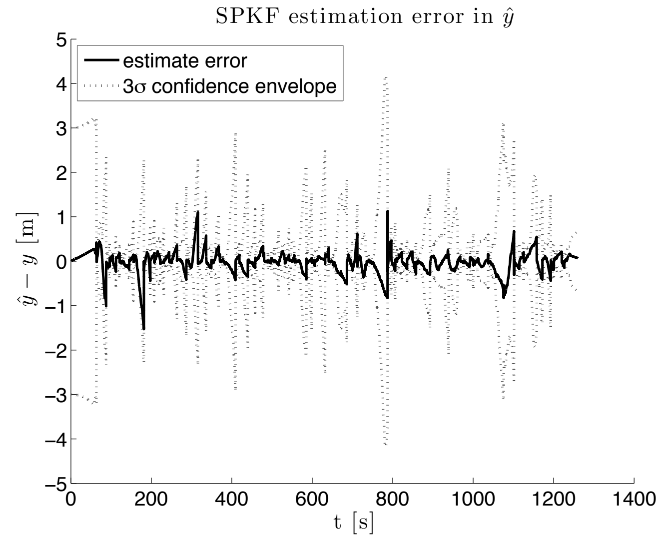
EKF estimation error in $\hat{y}$
estimate error
3σ confidence envelope
$\hat{y} - y$ [m]
t [s]

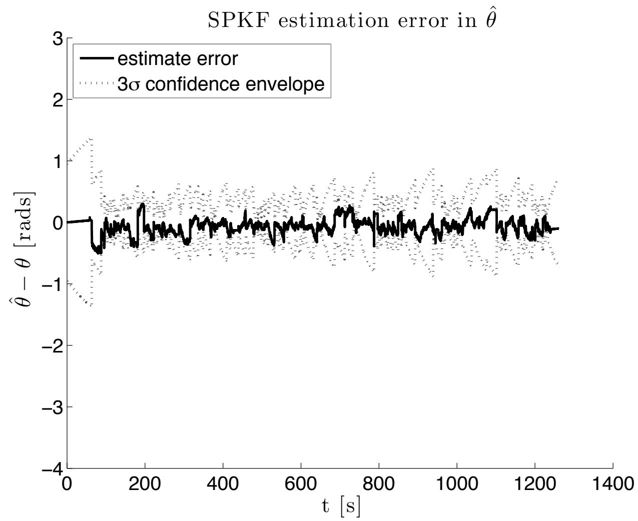


EKF estimation error in $\hat{\theta}$
estimate error
3σ confidence envelope
$\hat{\theta} - \theta$ [rads]
t [s]

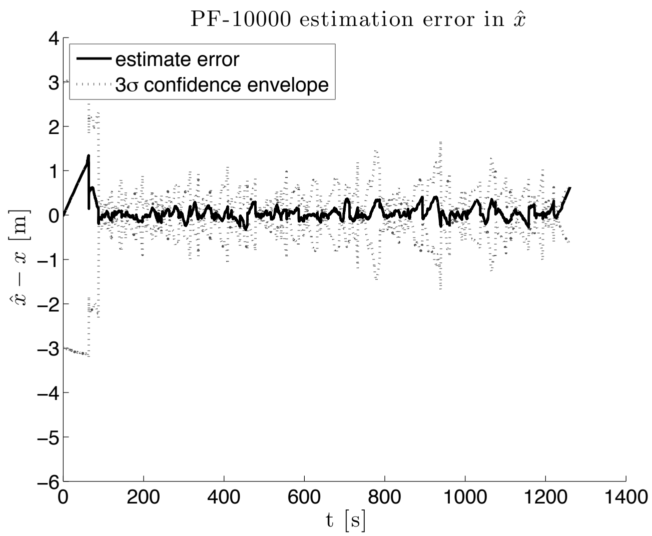
Results | SPKF

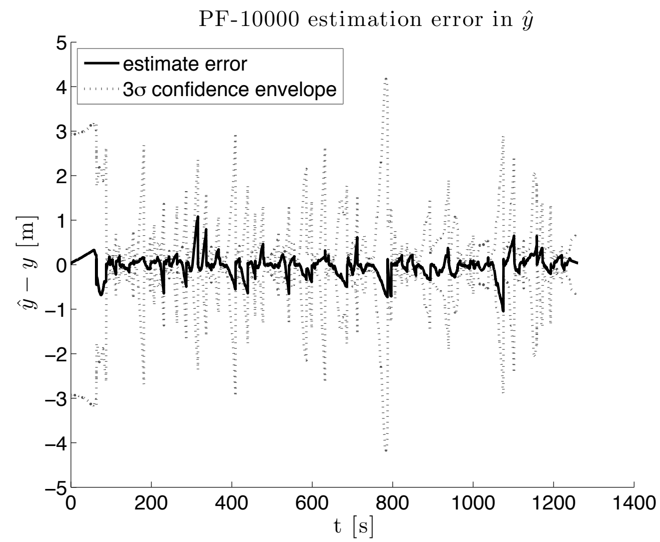


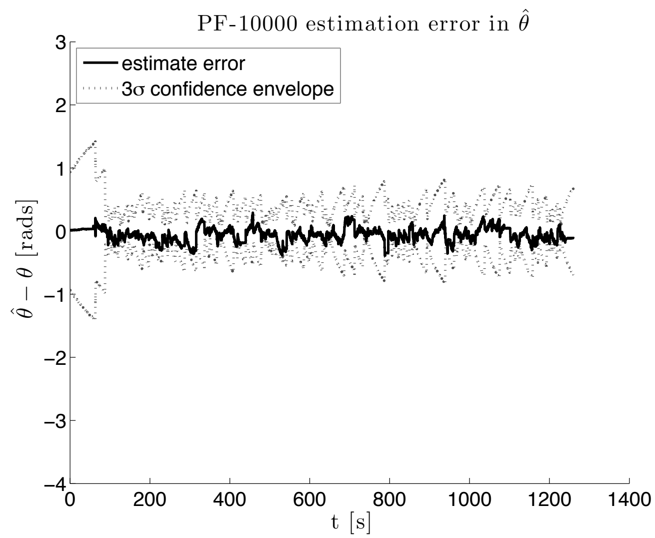




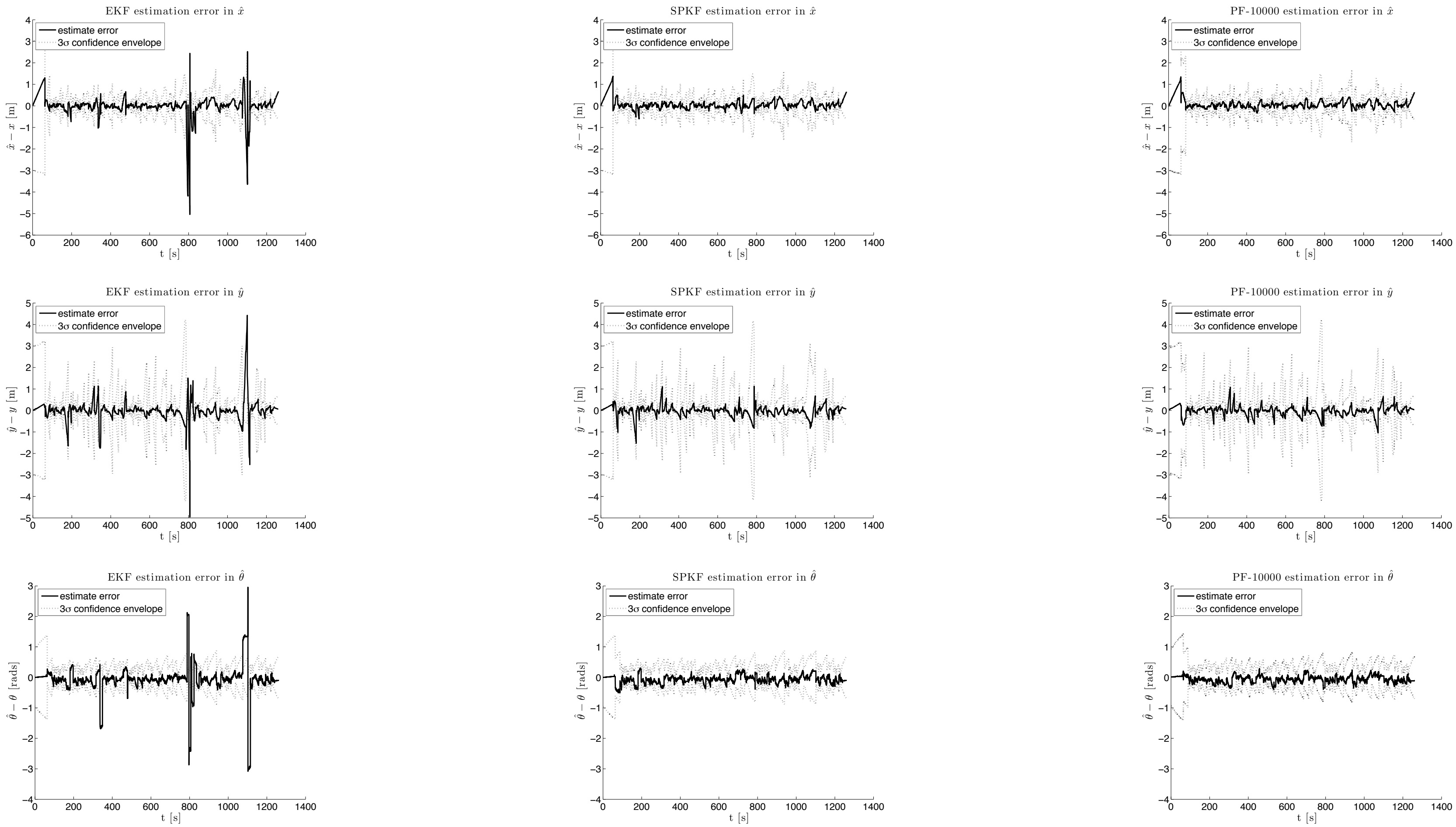
Results | PF


PF-10000 estimation error in $\hat{x}$
estimate error
3σ confidence envelope
$\hat{x} - x$ [m]
t [s]


PF-10000 estimation error in $\hat{y}$
estimate error
3σ confidence envelope
$\hat{y} - y$ [m]
t [s]


PF-10000 estimation error in $\hat{\theta}$
estimate error
3σ confidence envelope
$\hat{\theta} - \theta$ [rads]
t [s]

Results | EKF vs. SPKF vs. PF



EKF SPKF PF

Quantitative Results

(all RMS values)	EKF	SPKF	PF
$\hat{x} - x$ [m]	0.50	0.23	0.22
$\hat{y} - y$ [m]	0.48	0.24	0.22
$\hat{\theta} - \theta$ [rads]	0.51	0.14	0.12
$\hat{x} - \hat{x}_{pf}$ [m]	0.46	0.10	N/A
$\hat{y} - \hat{y}_{pf}$ [m]	0.42	0.21	N/A
$\hat{\theta} - \hat{\theta}_{pf}$ [rads]	0.50	0.12	N/A
$\hat{\sigma}_x - \hat{\sigma}_{x_{pf}}$ [m]	0.083	0.079	N/A
$\hat{\sigma}_y - \hat{\sigma}_{y_{pf}}$ [m]	0.093	0.095	N/A
$\hat{\sigma}_\theta - \hat{\sigma}_{\theta_{pf}}$ [rads]	0.028	0.023	N/A