# Introduction to Variational Methods in Imaging

Karteek Popuri

Department of Computing Science
University of Alberta

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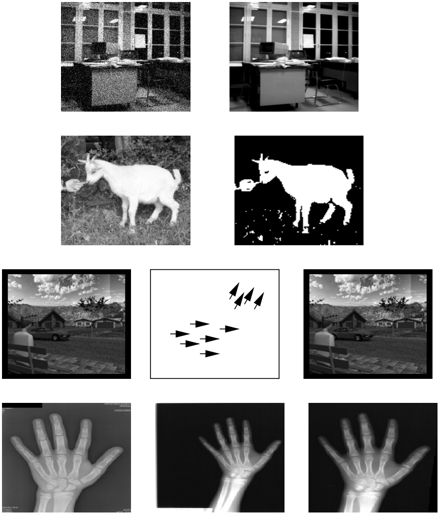
# Outline

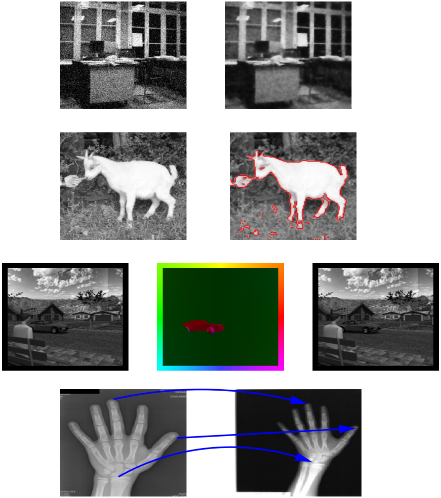
- Motivation
- Introduction to calculus of variations
- Numerical methods for variational minimization
- Applications of variational methods in imaging
- Conclusion
- Acknowledgments

# Common image processing tasks



- De-noising: Remove noise from an image
- Segmentation: Partition the image into object and background
- Optic flow: Estimate the apparent motion between two images
- Registration: Transform the source image to match the template image

# Image processing tasks as function estimation



- De-noising: Find a smooth approximation to the noisy image in the space of images
- Segmentation: Find a smooth closed curve between object and background
- Optic flow: Compute a smooth displacement field between two images
- Registration: Estimate a smooth and realistic deformation field that matches the corresponding points in template and source images

# Image processing tasks as function estimation

## General variational framework

- Goal: To determine an unknown function $\mathbf{u}(\mathbf{x})$ satisfying given constraints
- The constraints are formulated in the form of an energy functional as follows:

$$E[\mathbf{u}] = \int_{\Omega} \underbrace{D(\mathbf{u})}_{Data} + \underbrace{S(\mathbf{u})}_{Smoothing} + \underbrace{T(\mathbf{u})}_{Domain} \, d\mathbf{x}$$

- Using calculus of variations, determine the unknown function as the argument that minimizes the above energy:

$$\mathbf{u}^* = \underset{\mathbf{u} \in \mathcal{U}}{\operatorname{argmin}} \, E[\mathbf{u}]$$

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# Variational image de-noising

Consider the variational de-noising of some noisy image $I_0$, i.e., find the minimizer $I_\alpha$ of:

$$E[I] = \int_\Omega \underbrace{(I - I_0)^2}_{data} + \alpha \underbrace{\Psi(||\nabla u||^2)}_{smoothing} d\mathbf{x}$$

- In the above the, first term (data term, similarity term, fidelity term) encourages similarity to the original noisy image
- Second term (smoothness term, regularizer, penalizer) encodes the **smoothness constraint !**
- $\alpha > 0$ is the regularization parameter (smoothness weight)

# Variational image de-noising . . .

Now, we seek a minimizer of the functional $E[I]$:

$$I_\alpha = \underset{I \in \mathcal{I}}{\operatorname{argmin}} E[I] = \underset{I \in \mathcal{I}}{\operatorname{argmin}} \int_\Omega (I - I_0)^2 + \alpha\, \Psi(||\nabla u||^2) d\mathbf{x}$$

$I_\alpha$ then corresponds to the non-noisy or smoothed image

Calculus of variations gives the minimum of $E[I]$ as the solution of the Euler-Lagrange equation:

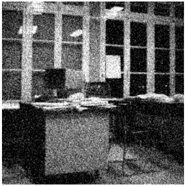
$$I - I_0 - \alpha\, \mathbf{div}(\Psi^{'}(||\nabla u||^2)\nabla u) = 0$$

$$\implies \frac{I - I_0}{\alpha} - \mathbf{div}(\Psi^{'}(||\nabla u||^2)\nabla u) = 0$$

This solved using gradient descent as:

$$\frac{\partial I}{\partial t} = \mathbf{div}(\Psi^{'}(||\nabla u||^2)\nabla u) - \frac{I - I_0}{\alpha}$$

# Variational image de-noising . . .



Noisy image

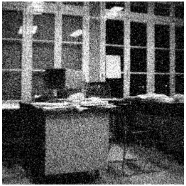
Perona-Malik ($\alpha = 5$)

Quadratic ($\alpha = 5$)

Common choices for the penalizer:

- Quadratic: $\Psi(s^2) = s^2$
- Perona-Malik [Perona et al., 1990]: $\Psi(s^2) = \lambda^2(\log(1 + \frac{s^2}{\lambda^2}))$
- Charbonnier [Charbonnier et al., 1994]: $\Psi(s^2) = 2\lambda^2\sqrt{1 + \frac{s^2}{\lambda^2}} - 2\lambda^2$
- Better (edge-preserving) smoothing by the Perona-Malik regularizer !

# Variational image de-noising . . .



Noisy image

Perona-Malik ($\alpha = 5$)

Quadratic ($\alpha = 5$)

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- Perona-
- Charbo $2\lambda^2$
- Better ( zer !

## What's next ?

- Other applications in imaging: Optic flow, Segmentation, Registration
- General framework for the numerical solution of variational minimization
- . . . **But first formal introduction to the calculus of variations**

# Functionals

- A functional is a correspondence that assigns a real number to each function belonging to a class
- The expression

$$E[y] = \int_{\Omega} F[\mathbf{x}, y(\mathbf{x}), \nabla y(\mathbf{x})]\, d\mathbf{x}$$

defines a functional $E[y]$, where $y(\mathbf{x}) \in \mathscr{D}_1(\Omega)$

- For example, we have already seen the functional for de-noising an image $I_0$, where:

$$F[I, \nabla I] = (I - I_0)^2 + \alpha\, \Psi(||\nabla I||^2)$$

# Minimization of functionals

Consider an increment $h(\mathbf{x})$ in the "independent" variable $y(\mathbf{x})$, we can then calculate the increment in the function $E[y]$ as:

$$\triangle E[y] = E[y+h] - E[y] = \int_\Omega [F(x, y+h, \nabla y + \nabla h) - F(x, y, \nabla y)]\, d\mathbf{x}$$

Using Taylor's theorem to expand the integrand, we obtain

$$\triangle E[y] = \int_\Omega [F_y(x, y, \nabla h)h - F_{\nabla y}(x, y, \nabla y)^T \nabla h]\, d\mathbf{x} + \mathcal{O}(h)$$

Ignoring the higher order terms and simplifying the notation we get the **first variation** of the above functional as:

$$\delta E[y] = \int_\Omega [F_y h - F_{\nabla y}^T \nabla h]\, d\mathbf{x}$$

# Minimization of functionals . . .

The necessary **integral** condition for the extremum is

$$\delta E[y] = \int_{\Omega} [F_y h - F_{\nabla y}^T \nabla h] \, d\mathbf{x} = 0 \quad \forall \, h \in \mathscr{D}_1(\Omega)$$

We then obtain the corresponding **Euler-Lagrange equation** as:

$$F_y - \mathbf{div}(F_{\nabla y}) = 0$$

This also gives rise to the so-called **natural (Neumann) boundary** conditions:

$$\mathbf{n}^T F_{\nabla y} = 0$$

(see pages 152 – 154 [Gelfand et al., 1963])

# Method 1: Gradient descent

- Set up a gradient descent evolution and discretize the resulting **parabolic** equation using Finite Differences (FD):

$$\frac{\partial y}{\partial t} = -(F_y - \mathbf{div}(F_{\nabla y}))$$

Using a time explicit scheme, in 2D for the above we have:

$$\frac{y_{i,j}^{(k+1)} - y_{i,j}^{(k)}}{\triangle t} = -(F_y - \mathbf{div}(F_{\nabla y}))_{i,j}^{(k)}$$

- results in a set of $N$ (number of grid points) linear equations in general
- the discretization of the grid is usually UNIFORM, i.e. for $i = \{1, 2, \ldots L\}$, $j = \{1, 2, \ldots W\}$ we have $x_{i+1,j} - x_{i,j} = \triangle_L$, $x_{i,j+1} - x_{i,j} = \triangle_W$

# Method 2: Time-lagged non-linearity

- Using Finite Differences (FD) discretize the **elliptic** equation:

$$(F_y - \mathbf{div}(F_{\nabla y}))_{i,j} = 0$$
$$\equiv \mathbf{a}_{i,j}^T(\mathbf{y})\mathbf{y} - b_{i,j}(\mathbf{y}) = 0$$

where $\mathbf{a}_{i,j}^T(.), b_{i,j}(.)$ can be non-linear $\quad \mathbf{y} = \{y_{i,j}\}, \mathbf{a} = \{a_{i,j}\}$

- results in a set of $N$ (number of grid points) non-linear equations
- the discretization of the grid is usually UNIFORM, i.e. for $i = \{1, 2, \ldots L\}$, $j = \{1, 2, \ldots W\}$ we have $x_{i+1,j} - x_{i,j} = \triangle_L$, $x_{i,j+1} - x_{i,j} = \triangle_W$

- Solve the above set of non-linear equations as a series of set of linear equations

- To obtain the current estimate $\mathbf{y}^{k+1}$ approximate the non-linear terms using the previous estimate $\mathbf{y}^k$ and obtain a linear system of the following form:

$$\mathbf{a}_{i,j}^T(\mathbf{y}_{i,j}^k)\mathbf{y}_{i,j}^{k+1} = b_{i,j}(\mathbf{y}_{i,j}^k)$$

# Method 3: Finite Element Method

- Solve integral extremum condition using the Finite Element Method (FEM)

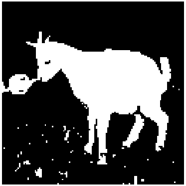
$$\delta E[y] = \int_{\Omega} [F_y h - F_{\nabla y}^T \nabla h] \, d\mathbf{x} = 0 \quad \forall\, h \in \mathscr{D}_1(\Omega)$$

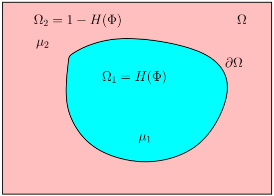
approximate $y$ using nodal basis functions:

$$y(\mathbf{x}) \approx \sum_{n=1}^{N} y(P_n)\phi_n(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \mathbb{R}^{L \times W}$$

  - Setting $h = \phi_i$, $i = \{1, 2, \ldots, N\}$, we get N linear equations
  - the discretization of the grid is usually NON-UNIFORM adapted to the problem domain and $N << L \times W$

# Segmentation



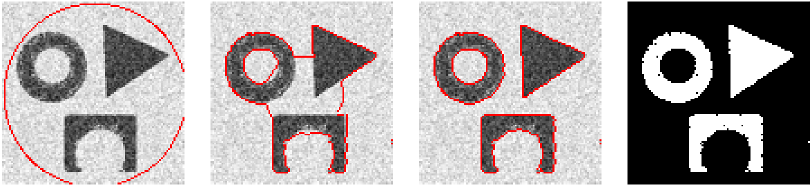



- Chan-Vese variational segmentation model [Chan et al., 2001]:

$$E[\Phi] = \int_\Omega \left( \underbrace{H(\Phi)(I - \mu_1)^2 + (1 - H(\Phi))(I - \mu_2)^2}_{data} + \nu \underbrace{||\nabla H(\Phi)||}_{smoothing} \right) d\mathbf{x}$$

- $\Phi$ is the level-set function, the segmentation boundary $\partial\Omega = \{\mathbf{x} \mid \Phi(\mathbf{x}) = 0\}$
- Separate the image domain into two regions of maximally distinct average intensities $(\mu_1, \mu_2)$ while keeping the boundary length $(||\nabla H(\Phi)||)$ small

# Segmentation . . .



Initialization  $t = 30$  $t = 100$

- The gradient descent evolution equation is given by:

$$\frac{\partial \Phi}{\partial t} = \delta(\Phi)\left((I-\mu_2)^2 - (I-\mu_1)^2 + \nu\,\mathbf{div}\left(\frac{\nabla\Phi}{||\nabla\Phi||}\right)\right)$$

- Region-based segmentation, NOT sensitive to initialization and noise
- Level-set representation easily handles topological changes

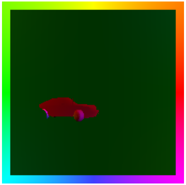
# Applications of variational methods in imaging (3)

## Optic flow

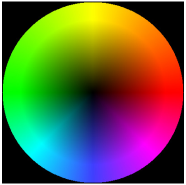
(*Images taken from* [Bruhn, 2006])

$I(x, y, t)$

$I(x+u, y+v, t+1)$



Displacement field



Color code

- Optic flow refers to the apparent motion of the scene between two consecutive image frames
- The goal is to compute the displacement field that maps the pixels in the first image to their new locations in the second image
- We assume brightness constancy and small displacements (**linearization**) for each pixel:

$$|I(x, y, t) - I(x+u, y+v, t+1)| \approx |I_x u + I_y v + I_t| = 0$$

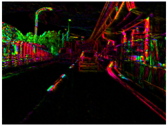
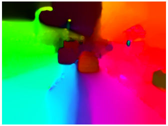
- 1 **equation** 2 **unknowns** $(u, v)$ at each pixel !

# Variational optic flow method

$$E[u,v] = \int_\Omega \underbrace{(I_x u + I_y v + I_t)^2}_{data} + \alpha \underbrace{\Psi(||\nabla u||^2 + ||\nabla v||^2)}_{smoothing} \, d\mathbf{x}$$

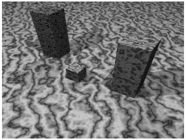
- Data term penalizes deviations from brightness constancy (**linearized**)
- Smoothness term penalizes deviations from a smooth flow field.
  - $\Psi(s^2) = s^2$ :**homogeneous** flow field [Horn and Schunck, 1981]
  - $\Psi(s^2) = \sqrt{s^2 + \epsilon^2}$ :**piecewise smooth** flow field [Schnörr, 1994]
- **Filling-in-effect**: in homogeneous regions WITHOUT edges, $I_x = I_y \approx 0$ $\implies I_x u + I_y v \approx I_t$, i.e NO contribution of data term w.r.t $(u, v)$, smoothing term propagates or "fills in" information from neighboring regions

(*Images taken from* [Bruhn, 2009])

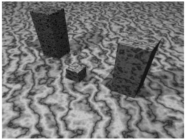
Edge information   Filling-in

# Variational optic flow method . . .

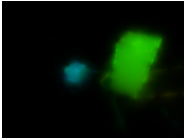
(*Images taken from* [Bruhn, 2009])



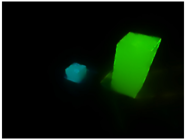
$I(x, y, t)$



$I(x+u, y+v, t+1)$



$\Psi(s^2) = s^2$
(Quadratic)



$\Psi(s^2) = \sqrt{s^2 + \epsilon^2}$
(Linear)

$$I_x^2 u + I_x I_y v + I_x I_t - \alpha \mathbf{div}(D \nabla u) = 0$$
$$I_x I_y u + I_y^2 v + I_y I_t - \alpha \mathbf{div}(D \nabla v) = 0$$

where $D = \Psi'(||\nabla u||^2 + ||\nabla v||^2)$

- 2 non-linear Euler-Lagrange equations
- Solved using the time-lagged non-linearity method

  - The linear penalizer prevents smoothing over flow edges and hence preserves discontinuities in the flow field

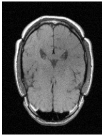
# Image registration

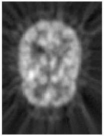
CT-CT

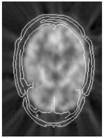


(*Images taken from* [Bruhn, 2009])

MRI-PET


Template


Source


Registered

(*Images taken from* [Du et al., 2006])

- Similar to optic flow, estimate a realistic displacement (deformation) field mapping corresponding pixels in the template image to the source image
- The source image is **warped** (based on the deformation field) using interpolation to obtain the registered image
- **Challenges: Large displacements, images from different modalities, need realistic regularizers (smoothing terms)**

# Image registration . . .

(*Images taken from* [Heldmann et al., 2004])

Template (T1) Source (T2) Registered

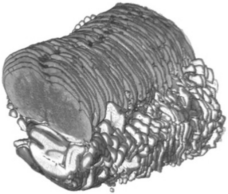
$$E[u,v] = -\int_{[0,255]^2} \underbrace{p^{[I_1,\hat{I}_2]} \log \frac{p^{[I_1,\hat{I}_2]}}{p^{[I_1]}\,p^{[\hat{I}_2]}}}_{data}\, da\, db + \alpha \int_{\Omega} \underbrace{\left(||\nabla u||^2 + ||\nabla v||^2\right)}_{smoothing} d\mathbf{x}$$

$$p \equiv p(a,b)$$

- Mutual information is used to compute the similarity between the two different image modalities (data term NOT linearized)
- Solved using lagged non-linearity with image warping at each step, i.e $\hat{I}_2^{(k)} = I_2(x + u^{(k)}, y + v^{(k)})$

# Image registration . . .

(*Images taken from* [Wirtz et al., 2004])



Before registration

After registration

$$E[u,v] = \int_\Omega \underbrace{(I_1 - \hat{I}_2)^2}_{data} + \mu \underbrace{\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)}_{smoothing} + (\lambda+\mu)\underbrace{(u_x + v_y)^2}_{elasticity} dx\, dy$$

where $\hat{I}_2 = I_2(x+u, y+v)$

- Additional smoothing term based on elasticity theory, i.e., $\textbf{div}\binom{u}{v} = 0$ (no sources or sinks)

# Conclusions

- Many image processing tasks can be posed as variational problems, which can then be solved in a common energy minimization framework
- The variational formulations in general can be easily extended (modified) to incorporate additional (different) set of constraints
- Efficient numerical techniques (both FD-based, FEM-based) exist that can provide a fast and an accurate solution to the variational minimization
- Other applications in imaging and vision include stereo, structure from motion, shape estimation

- Supervisors Dana Cobzas and Martin Jägersand
- Members of the Computer Vision group: Neil Birkbeck, David Lovi, Parisa Mosayebi, Azad Shademan, Eric Coulthard at the University of Alberta
- Joachim Weickert, Andres Bruhn, Mathematical Image Analysis (MIA), University of Saarbruecken, Germany

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